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Instanton counting and dielectric branes

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Abstract: We consider the Hanany-Witten type brane configuration in a background of RR 4-form field strength and examine the behavior of Euclidean D0-branes propagating between two NS5-branes. We evaluate the partition function of the D0-branes and show that it coincides with the Nekrasov partition function of instantons for four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. In this analysis, the Myers effect plays a crucial role. We apply the same method to the brane configuration realizing four-dimensional $\mathcal{N} = 2$ theory with hypermultiplets in the fundamental representation and reproduce the corresponding Nekrasov partition function.

Keywords: Brane Dynamics in Gauge Theories, Solitons Monopoles and Instantons, Supersymmetric gauge theory, D-branes.

Contents

1. Introduction

Supersymmetric gauge theory is one of the most exciting topics in high energy physics from various points of view. An important property of supersymmetric theory is that we can exactly deal with the theory using the algebra of supersymmetry. Among such exact results, one of the most significant steps toward understanding the non-perturbative property of (supersymmetric) gauge theory is that Seiberg and Witten exactly determined the low energy prepotential of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory [1, 2]. Furthermore, the instanton contribution to the prepotential of the $\mathcal{N} = 2$ theory has been exactly calculated through a partition function of Young diagrams (the Nekrasov partition function) [3, 4]. This partition function is obtained by explicitly carrying out the path-integral of four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory deformed by a constant graviphoton background (Ω-background) parametrized by ϵ.[1] Thanks to this deformation, we can carry out the integration over the instanton moduli space using the so-called localization technique [5 – 8], which yields a partition function of Young diagrams (see [9] for further development). In particular, the leading term of the free energy in

[1]The Ω-background is parametrized by two parameters ϵ_1 and ϵ_2 in general. In this paper, however, we concentrate on the case of $\epsilon_1 = -\epsilon_2 = \epsilon$.

the expansion in powers of ϵ has been shown to coincide with the low energy prepotential of four-dimensional $\mathcal{N} = 2$ SU(N) supersymmetric Yang-Mills theory [10 − 12] (see also [13]). Thus the k-instanton contribution to the prepotential can be explicitly read off by evaluating the partition function.

The Nekrasov partition function and the nature of the Ω-background have been intensively studied in terms of string theory. In general, four-dimensional $\mathcal{N} = 2$ gauge theory can be realized in Type IIB superstring theory as an effective theory on fractional D3-branes, where instanton effects come from D(-1)-branes bound to the fractional D3-branes. In [14], it was shown that the effective action of the fractional D3-D(-1) system in a RR-background coincides with the instanton effective action of four-dimensional $\mathcal{N} = 2$ theory in the Ω-background. This means the RR-background is equivalent with the Ω-background in this brane configuration. This brane configuration in the RR-background and the deformed four-dimensional gauge theory have been further studied in [15 − 20]. For a relation between the Nekrasov formula and topological string, see [21 − 26]. In this connection, relations with topological vertex [27 − 29], melting crystal [30, 31], and integrable systems have been studied in detail [32 − 45].[2]

The purpose of this paper is to reproduce the Nekrasov partition function in terms of the Hanany-Witten type brane configuration in Type IIA superstring theory [49, 50], that is, a system of N D4-branes stretched between two parallel NS5-branes. This configuration is a T-dual of the system of fractional D3-branes mentioned above and the instanton effects of four-dimensional Yang-Mills theory come from (Euclidean) D0-branes "propagating" between the two NS5-branes. We explicitly show that we can identify the Ω-background as a background of RR 4-form field strength in this brane configuration as expected. In general, D0-branes behave as a dielectric D2-brane in a background of constant RR 4-form field strength [51]. In addition, a D2-brane can have the end on a Type IIA NS5-brane whose boundary is coupled with the self-dual 2-form potential in the world-volume of the NS5-brane [52]. We evaluate the potential energy coming from the interaction between the boundaries of the dielectric D2-branes in the NS5-branes and the "kinetic energy" of the D0-branes propagating between the two NS5-branes. We show that the partition function of these configurations coincides with the Nekrasov partition function of instantons. We also reproduce the Nekrasov partition function for $\mathcal{N} = 2$ theory with hypermultiplets in the fundamental representation using the same method.

The organization of this paper is as follows. In the next section, after briefly reviewing the Hanany-Witten type brane configuration, we show that the Ω-background is identical with a background of RR 4-form field strength in this brane configuration. In section 3, we analyze the behavior of D0-branes in detail and reproduce the Nekrasov partition function as a partition function of the D0-branes. In section 4, we reproduce the instanton partition function of $\mathcal{N} = 2$ theory with hypermultiplets in the fundamental representation using the same method developed in section 3. Section 5 is devoted to conclusion and discussion. In appendix A, we review the Frobenius representation of Young diagram. In appendix B, we

[2]For relations between the Nekrasov partition function and simple physical systems like 2D Yang-Mills theory, 3D Chern-Simons theory and matrix models, see [46 − 48].

summarize the Nekrasov partition function and rewrite it in the Frobenius representation. In appendix C, we summarize the Nekrasov formula for $\mathcal{N}=2$ theory with hypermultiplets in the fundamental representation.

2. Brane configuration in a RR-background

In this section, we briefly review the Hanany-Witten type brane configuration to realize four-dimensional $\mathcal{N}=2$ supersymmetric gauge theory. We introduce NSNS B-field and RR 4-form field strength in the background and show that the Myers term introduced in the effective potential of D0-branes is identical with the deformation of instanton effective action by the Ω-background.

2.1 Brane configuration

In order to realize four-dimensional $\mathcal{N}=2$ supersymmetric SU(N) Yang-Mills theory, we consider a system of two parallel NS5-branes and N D4-branes stretching between the NS5-branes [50] (see also the review [53] and references therein). The world-volumes of the NS5-branes and the D4-branes are along the 012345 and 01236 directions, respectively, and the $\mathcal{N}=2$ gauge theory arises as the low energy effective theory on the common directions 0123. We introduce complex combinations of the coordinates as

$$z_1 \equiv x^0 + ix^1, \quad z_2 \equiv x^2 + ix^3, \quad v \equiv x^4 + ix^5, \tag{2.1}$$

and rename x^6 as τ as well. We assume that the NS5-branes sit at $\tau=0$ and L, respectively, and the D4-branes stretch between NS5-branes at $v=a_l$ ($l=1,\ldots,N$). Since we are interested in instanton contributions to the four-dimensional gauge theory, we also add (Euclidean) D0-branes bound to the D4-branes that propagate between the NS5-branes [4] (see figure 1). From the four-dimensional theory point of view, a_l correspond to the classical vev of the adjoint scalar fields and the coupling constant of the gauge theory is related to L as

$$\frac{1}{g_{\rm YM}^2} = \frac{L}{g_s l_s}, \tag{2.2}$$

where g_s and l_s are the string coupling constant and the string length, respectively.

Precisely speaking, the definition of L is ambiguous since the D4-branes deform the world-volume of the NS5-branes as [50]

$$\tau = \pm g_s l_s \sum_{l=1}^{N} \left(\log \frac{|v-a_l|}{\Lambda} \right), \tag{2.3}$$

where Λ is a positive constant. From the relation (2.2), the gauge coupling constant is a function of v. In the region $v \gg a_l$, it gives

$$\frac{1}{g_{\rm YM}^2(v)} \sim \log \frac{|v|^{2N}}{\Lambda^{2N}}. \tag{2.4}$$

Thus it is reasonable to interpret Λ as the dynamical scale of the four-dimensional gauge theory and define L as the distance between the NS5-branes at the cut-off scale.

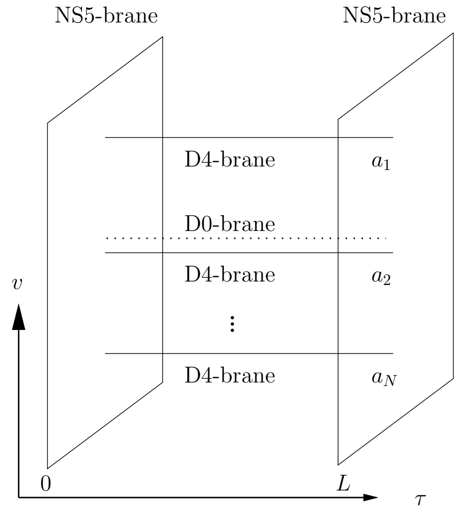

Figure 1: The brane configuration in $(\tau, v, \bar{v})$ space. The NS5-branes sit at $\tau = 0$ and $\tau = L$, and the D4-branes are expressed as solid lines stretch between the NS5-branes at $v = a_1, \ldots, a_N$. We also consider Euclidean D0-branes propagating between the NS5-branes, which is expressed as a dotted line in the figure.

Under this configuration, in addition to the coupling between light modes on D4-branes, there are couplings of these light modes to bulk gravity fields, to fields living on NS5-branes, and to massive modes on D4-branes in general. Since we are interested in gauge theory dynamics, these coupling must be small. To do so, we take the limit,

$$g_s \to 0, \quad L/l_s \to 0. \tag{2.5}$$

Furthermore, in order for the system to be consistent with the interpretation of Coulomb branch of the four-dimensional gauge theory, the vev of the adjoint scalar must be sufficiently smaller than the Kaluza-Klein scale $1/L$. Therefore, a_l must satisfy

$$a_{ln} \equiv a_l - a_n \ll l_s^2/L. \tag{2.6}$$

We must also require

$$a_{ln} \ll l_s, \quad a_{ln} \ll L, \tag{2.7}$$

in order to decouple massive modes of open strings on D4-branes.

2.2 Effective action of D0-branes and its deformation by flux

Suppose that k D0-branes are bound to D4-branes and let us consider the low energy effective theory on the D0-branes. The degrees of freedom of the effective theory come from open strings between D0-branes and those between D0-branes and D4-branes. The former gives $k \times k$ bosonic complex matrices B_1, B_2 and ϕ, which are collective coordinates corresponding to the directions z_1, z_2 and v, respectively, and fermionic complex matrices Ψ_{B_1}, Ψ_{B_2} and η with the same size. The latter gives bosonic complex matrices I and J with the size $k \times N$ and $N \times k$, respectively, and fermionic complex matrices Ψ_I and Ψ_J with the size $k \times N$ and $N \times k$, respectively.

The effective theory is a quantum mechanics of these matrix variables [54, 55]. Since we are interested in BPS configuration of the system, however, it is sufficient to look at the potential terms, which is efficiently expressed as a BRST exact form as

$$V = Q\mathrm{Tr}\,_k\left(\chi_r\mu_r + \chi_c\mu_c + \chi_c^\dagger\mu_c^\dagger + \bar{\phi}\mu_\eta\right), \tag{2.8}$$

with

$$\begin{aligned}
\mu_r &\equiv [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J,\\
\mu_c &\equiv [B_1, B_2] + IJ,\\
\mu_\eta &= [B_1, \Psi_{B_1}^\dagger] - [B_1^\dagger, \Psi_{B_1}] + [B_2, \Psi_{B_2}^\dagger] - [B_2^\dagger, \Psi_{B_2}] + \Psi_I I^\dagger + I\Psi_I^\dagger - J^\dagger\Psi_J - \Psi_J^\dagger J,
\end{aligned} \tag{2.9}$$

where we have introduced auxiliary fermionic matrices χ_r and χ_c and their superpartners H_r and H_c with the size $k \times k$. The BRST transformation is given by

$$\begin{aligned}
QB_i &= \Psi_{B_i}, & Q\Psi_{B_i} &= [\phi, B_i], \qquad (i = 1, 2)\\
QI &= \Psi_I, & Q\Psi_I &= \Phi I - Ia,\\
QJ &= \Psi_J, & Q\Psi_J &= -J\phi + aJ,\\
Q\chi_r &= H_r, & QH_r &= [\phi, \chi_r],\\
Q\chi_c &= H_c, & QH_c &= [\phi, \chi_c],\\
Q\bar{\phi} &= \eta, & Q\eta &= [\phi, \bar{\phi}], \qquad Q\phi = 0,
\end{aligned} \tag{2.10}$$

where $a \equiv \mathrm{diag}(a_1, \ldots, a_N)$. As a nature of a system of k D0-branes, the effective theory possesses gauge symmetry $\mathrm{U}(k)$. In fact, (2.8) is invariant under the transformation,

$$\begin{aligned}
&B_i \to gB_ig^{-1}, && \phi \to g\phi g^{-1}, && I \to gI, && J \to Jg^{-1},\\
&\Psi_{B_1} \to g\Psi_{B_i}g^{-1}, && \eta \to g\eta g^{-1}, && \Psi_I \to g\Psi_I, && \Psi_J \to \Psi_J g^{-1},
\end{aligned} \tag{2.11}$$

with $g \in \mathrm{U}(k)$. Note that the potential (2.8) is nothing but the instanton effective action of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory. Indeed the BPS configuration of D0-branes, that is, the instanton moduli space is determined by solving the ADHM equations,

$$\mu_r = 0, \qquad \mu_c = 0, \tag{2.12}$$

as well as

$$[\phi, B_i] = 0, \quad \phi I - Ia = 0, \quad -J\phi + aJ = 0, \quad [\phi, \bar{\phi}] = 0. \tag{2.13}$$

For more detail, see [56] and references therein.

We deform the effective potential (2.8) by introducing flux in the background of the brane configuration. We first introduce a constant NSNS B-field,

$$B^{(2)} = \frac{\zeta}{2}\left(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2\right). \qquad (\zeta > 0) \tag{2.14}$$

It is familiar that NSNS B-field background introduces noncommutativity into the world-volume of D4-branes [57];

$$[z_1, \bar{z}_1] = \frac{\zeta}{2}, \quad [z_2, \bar{z}_2] = \frac{\zeta}{2}, \tag{2.15}$$

which modifies μ_r in the potential (2.8) as

$$\mu_r \to \mu_r - \zeta. \tag{2.16}$$

By this modification, it turns out that the size moduli of instantons cannot be zero, that is, the small instanton singularity of the moduli space is resolved [58, 59]. From the D0-brane point of view, the presence of NSNS B-field prevents D0-branes to escape from D4-branes without breaking supersymmetry [57].

In addition to the NSNS B-field, we further introduce a RR 3-form potential,

$$C^{(3)} = \epsilon\,(v + \bar{v})\,(-d\tau \wedge dz_1 \wedge d\bar{z}_1 + d\tau \wedge dz_2 \wedge d\bar{z}_2)\,, \tag{2.17}$$

or, equivalently, a RR 4-form field strength,

$$F^{(4)} = 2\epsilon\left(d\tau \wedge dx^4 \wedge dz_1 \wedge d\bar{z}_1 - d\tau \wedge dx^4 \wedge dz_2 \wedge d\bar{z}_2\right). \tag{2.18}$$

As shown in [51], D0-branes have a coupling with RR 4-form field strength through the so-called Myers term,

$$F^{(4)}_{\tau ijk}\,\mathrm{Tr}\left(\Phi^i[\Phi^j, \Phi^k]\right), \tag{2.19}$$

where $F^{(4)}$ is the 4-form field strength and Φ^i are the collective coordinates of D0-branes. From (2.18) and (2.19), we see that the Myers term,

$$\epsilon\,\mathrm{Tr}_k\left\{(\phi + \bar{\phi})\left([B_1, B_1^\dagger] - [B_2, B_2^\dagger]\right)\right\}, \tag{2.20}$$

is added to the effective potential (2.8) due to the RR 4-form field strength (2.18). It is easy to show that this modification is achieved by modifying the BRST charge Q in (2.8) to Q_ϵ defined by

$$\begin{aligned} Q_\epsilon \Psi_{B_1} &= [\phi, B_1] + \epsilon B_1, \\ Q_\epsilon \Psi_{B_2} &= [\phi, B_2] - \epsilon B_2, \end{aligned} \tag{2.21}$$

and the others are the same with (2.10). This is exactly the same deformation of the instanton effective action in the Ω-background [3]. Thus, as expected, we can conclude that the Ω-background is equivalent with the background of RR 4-form field strength (2.18) in the brane configuration given above.

By combining both the effects of NSNS B-field (2.14) and RR 3-form (2.17), we obtain the deformed D0-brane effective potential;

$$V_{\mathrm{mod}} = Q_\epsilon \mathrm{Tr}\left(\chi_r(\mu_r - \zeta) + \chi_c \mu_c + \chi_c^\dagger \mu_c^\dagger + \bar{\phi}\mu_\eta\right), \tag{2.22}$$

with (2.9) and (2.21).

3. Nekrasov partition function from D0-branes

The instanton part of the Nekrasov partition function (B.3) is originally obtained by explicitly evaluating the integral,

$$Z_k(\epsilon, \Lambda) = \int [dB_1 dB_2 \cdots] e^{-V_{\mathrm{mod}}}, \tag{3.1}$$

by using a property that the integral is localized at Q_ϵ-invariant points in the moduli space of instantons [7−9]. In the language of brane configuration, it corresponds to counting the BPS configurations of D0-D4 bound state in the background of the NSNS B-field (2.14) and the RR 4-form field strength (2.18). Such a configuration is obtained by solving the equations,

$$\begin{aligned} [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta, \qquad & [B_1, B_2] + IJ = 0, \\ [B_1, \phi] = \epsilon B_1, \qquad & [B_2, \phi] = -\epsilon B_2, \qquad [\phi, \bar{\phi}] = 0, \\ \phi I - Ia = 0, \qquad & J\phi - aJ = 0. \end{aligned} \tag{3.2}$$

In this section, we reproduce the instanton part of the Nekrasov partition function as a partition function of the D0-branes in the brane configuration introduced in the previous section. For simplicity, we start with the case of $N = 1$ and the case of $N > 1$ follows that.

3.1 $N = 1$

Although we can set $a = 0$ in this case, we keep it for a later discussion. It is easy to show that the matrices,

$$B_1^{(\tilde{n},n;a)} \equiv \begin{pmatrix} 0 & \sqrt{\zeta} & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & \sqrt{\tilde{n}\zeta} & & & \\ & & & 0 & 0 & & \\ & & & & 0 & \ddots & \\ & & & & & \ddots & 0 \\ & & & & & & 0 \end{pmatrix}, \qquad B_2^{(\tilde{n},n;a)} = \begin{pmatrix} 0 & & & & & & \\ 0 & \ddots & & & & & \\ & \ddots & 0 & & & & \\ & & 0 & 0 & & & \\ & & & \sqrt{n\zeta} & 0 & & \\ & & & & \ddots & \ddots & \\ & & & & & \sqrt{\zeta} & 0 \end{pmatrix},$$

$$\phi^{(\tilde{n},n;a)} \equiv \begin{pmatrix} a - \tilde{n}\epsilon & & & & & & \\ & \ddots & & & & & \\ & & a - \epsilon & & & & \\ & & & a & & & \\ & & & & a + \epsilon & & \\ & & & & & \ddots & \\ & & & & & & a + n\epsilon \end{pmatrix}, \qquad I^{(\tilde{n},n;a)} \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{(n + \tilde{n} + 1)\zeta} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$J^{(\tilde{n},n;a)} \equiv 0, \tag{3.3}$$

solve the equations (3.2) for any $\tilde{n}, n \in \mathbb{Z}_{\geq 0}$ with $\tilde{n} + n + 1 = k$. We call (3.3) as an irreducible solution. The equations (3.2) can be generally solved as block diagonal matrices that consist of irreducible solutions;

$$
\begin{aligned}
B_i^{(\tilde{\mathbf{n}},\mathbf{n};a)} &\equiv \begin{pmatrix} B_i^{(\tilde{n}_1,n_1;a)} & & \\ & \ddots & \\ & & B_i^{(\tilde{n}_L,n_L;a)} \end{pmatrix}, \qquad & \phi^{(\tilde{\mathbf{n}},\mathbf{n};a)} &\equiv \begin{pmatrix} \phi_i^{(\tilde{n}_1,n_1;a)} & & \\ & \ddots & \\ & & \phi_i^{(\tilde{n}_L,n_L;a)} \end{pmatrix}, \\
I^{(\tilde{\mathbf{n}},\mathbf{n};a)} &\equiv \begin{pmatrix} I^{(\tilde{n}_1,n_1;a)} \\ \vdots \\ I^{(\tilde{n}_L,n_L;a)} \end{pmatrix}, & J^{(\tilde{\mathbf{n}},\mathbf{n};a)} &\equiv 0,
\end{aligned} \tag{3.4}
$$

up to the gauge transformation (2.11) and/or a discrete transformation that keeps the potential (2.22) invariant. Here we have defined $\tilde{\mathbf{n}}$ and $\mathbf{n}$ as $\tilde{\mathbf{n}} = (\tilde{n}_1, \ldots, \tilde{n}_L)$ and $\mathbf{n} = (n_1, \ldots, n_L)$ with satisfying

$$
\sum_{i=1}^{L} (\tilde{n}_i + n_i + 1) \equiv \sum_{i=1}^{L} k_i = k, \tag{3.5}
$$

and we call this solution as a reducible solution in the following. Note that, by using the center Z_k of the gauge group $\mathrm{U}(k)$, we can always rearrange $\tilde{\mathbf{n}}$ and $\mathbf{n}$ so that they satisfy

$$
\tilde{n}_1 \geq \cdots \geq \tilde{n}_L \geq 0, \qquad n_1 \geq \cdots \geq n_L \geq 0. \tag{3.6}
$$

What kind of configuration does the solution (3.4) express? In order to answer this question, let us first recall the situation of D0-D4 bound state in a constant self-dual NSNS B-field background. From the view point of D4-brane, D0-branes behave as instantons of four-dimensional gauge theory whose size is bounded from below as a result of non-commutativity introduced by the B-field. As a result, the D0-branes are smeared into the four-dimensional (Euclidean) world-volume of the D4-branes with the size $\sqrt{|B|}$ [58]. On the other hand, consider D0-branes in a constant RR 4-form field strength background. In this case, the D0-branes expand into a fuzzy two-sphere, which can also be regarded as a spherical D2-brane to which D0-branes are bound [51]. Combining them, we can think of the irreducible solution (3.3) as a spherical configuration of D0-branes that is smeared in the four-dimensional space along (z_1, z_2) and is bulging from the world-volume of D4-brane in the x^4-direction. Looking at the solution (3.3), the typical size of this configuration along (z_1, z_2) is $\mathcal{O}(\sqrt{\zeta})$ and that along the x^4-direction is $\mathcal{O}(\epsilon)$ as expected. This configuration can be also interpreted as a spherical D2-branes to which D0-branes are bound as mentioned above. From the irreducible solution $\phi^{(\tilde{n},n;a)}$, we see that the D0-branes locate at $x^4 = a - \tilde{n}\epsilon, a - (\tilde{n}-1)\epsilon, \ldots, a + n\epsilon$. Since the D0-branes are bound to a spherical D2-brane, it is reasonable to think that the D2-brane cuts the x^4-axis at $x^4 = -(\tilde{n} + 1/2)\epsilon$ and $x^4 = (n + 1/2)\epsilon$ (figure 2). We can give the same interpretation for the reducible solution (3.4); it would express a set of L spherical D2-branes that cut the x^4-axis at $x^4 = a - (\tilde{n}_i + 1/2)\epsilon$ and $x^4 = a + (n_i + 1/2)\epsilon$ to which D0-branes are bound at

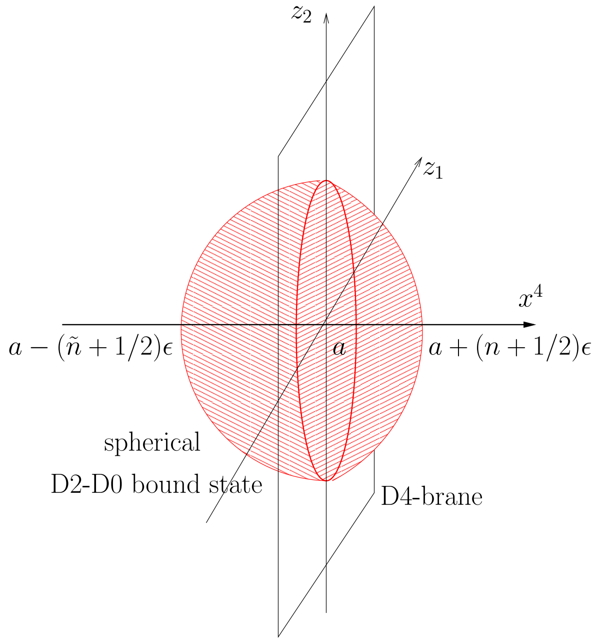

Figure 2: The spherical brane configuration corresponding to the irreducible solution (3.3). We can interpret this configuration as a fuzzy distribution of D0-branes or a spherical D2-brane to which D0-branes are bound. In the latter interpretation, the D0-branes locate at $x^4 = a - \tilde{n}\epsilon, a - (\tilde{n} - 1)\epsilon, \ldots, a + n\epsilon$ in the spherical D2-brane, which cuts the x^4-axis at $x^4 = a - (\tilde{n} + 1/2)\epsilon$ and $x^4 = a + (n + 1/2)\epsilon$.

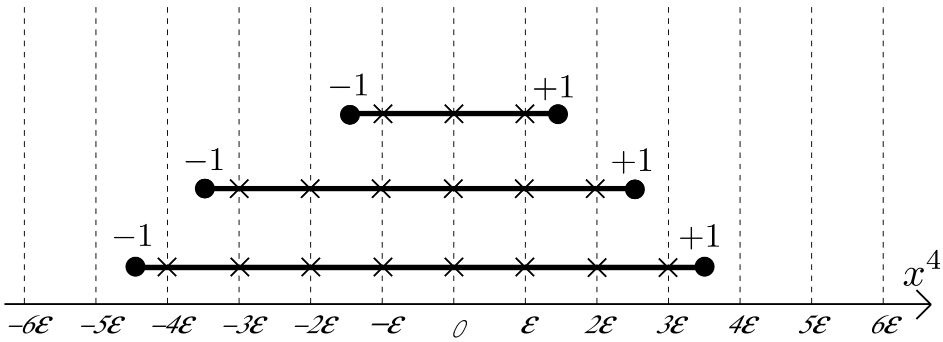

Figure 3: The boundary of the spherical D2-branes in the v-plane of the NS5-brane. We have depicted the case of $(\tilde{\mathbf{n}}|\mathbf{n}) = (4, 3, 1|3, 2, 1)$ and $a = 0$ as an example. In the v-plane, the boundary of each spherical brane can be seen as a 1-dimensional object. The edges of the object have opposite charges that make Coulomb potential in the v-plane. The crosses on the objects express the positions of D0-branes. Although we have drawn each objects as if they were shifted in some direction, they are overlapping on the x^4-axis in reality.

$x^4 = a - \tilde{n}_i\epsilon, \ldots, a + n_i\epsilon$ $(i = 1, \ldots, L)$.[3] Note that the net D2-brane charge of this system is zero. This corresponds to the observation that we need two patches to make a 2-sphere, which are regarded as a D2-brane and an anti-D2-brane, respectively.

[3]This is essentially the same configuration obtained in [16].

Since we are interested in dynamics of four-dimensional gauge theory and ϵ must be much smaller than a_{ln} (in the case of $N > 1$), ϵ and ζ should satisfy,

$$\epsilon \ll \sqrt{\zeta}, \tag{3.7}$$

from the requirement of (2.6) and (2.7). In this limit, each spherical D2-brane behaves as a pair of D2-brane and anti-D2-brane at $x^4 = -(\tilde{n}_i + 1/2)$ and $x^4 = n_i + 1/2$, respectively, which are smeared in the four-dimensional space (z_1, z_2). In general, a D2-brane can end on a Type IIA NS5-brane and the boundary behaves as a "string" coupled with self-dual 2-form potential in the world-volume of NS5-brane [52]. Recalling that the D0-branes propagate between the NS5-branes and the spherical D2-branes are smeared in a four-dimensional space (z_1, z_2), the boundaries of the D2-branes and anti-D2-branes are also smeared in (z_1, z_2), that would create a two-dimensional Coulomb potential in the v-plane that is transverse to (z_1, z_2) in each of the two NS5-branes. Therefore, looking at the v-plane in one of the NS5-branes, the boundary of each spherical brane can be seen as a 1-dimensional object spanned from $x^4 = a - (\tilde{n}_i + 1/2)\epsilon$ to $x^4 = a + (n_i + 1/2)\epsilon$ whose edges have opposite charges ± 1 for the two-dimensional Coulomb force as long as (3.7) is satisfied. We depict an example of the configuration in the v-plane in figure 3.

Now let us estimate the energy of the system corresponding to the solution (3.4) coming from the boundaries. We must take into account (1) the Coulomb force between charges in the NS5-branes and (2) the "kinetic energy" of D0-branes propagating between the NS5-branes:

(1) Coulomb force between charges in the NS5-branes.

In each NS5-brane, we assign charges -1 and $+1$ to the edges at $v = a - \epsilon(\tilde{n}_i + 1/2)$ and $v = a + \epsilon(n_i + 1/2)$ $(i = 1, \ldots, L)$, respectively.[4] The potential energy created by the two-dimensional Coulomb force between a charge $q = \pm 1$ at $v = x$ and a charge $q' = \pm 1$ at $v = y$ can be written as[5]

$$V(x, q; y, q') = -qq' \log |x - y| \,. \tag{3.8}$$

Therefore, the potential energies between the edges of the 1-dimensional objects can be estimated as

$$V_1 = 2 \log\left(\epsilon^L \frac{\prod_{i,j=1}^{L} |n_i + \tilde{n}_j + 1|}{\prod_{i<j}^{L} |n_i - n_j|\, |\tilde{n}_i - \tilde{n}_j|} \right), \tag{3.9}$$

where the factor 2 comes from the same effect from the two NS5-branes.

(2) D0-branes propagating between the NS5-branes.

We estimate the kinetic energy of each spherical brane by regarding that k_i D0-branes at $x^4 = a - \tilde{n}_i\epsilon, \ldots, a + n_i\epsilon$ are propagating from one NS5-brane to the other NS5-brane.

[4]Precisely speaking, the charges of the edges are opposite in each of the NS5-branes. But we do not need to distinguish them since the NS5-branes are separated with each other.

[5]Considering a M5-brane on which a D2-brane ends in M-theory, we can estimate the coefficient of (3.8) using the M5 and M2 charges $\mu_{\rm M5}$ and $\mu_{\rm M2}$ as $\frac{\mu_{\rm M2}^2}{2\pi\mu_{\rm M5}} = 1$.

From (2.3), we see that the distance between the NS5-branes at the position v is given by

$$d(v) = 2g_s l_s \left(\log \frac{|v-a|}{\Lambda} - \delta_{v,a} \log 0 \right), \tag{3.10}$$

where the second term is necessary to regularize the distance at $v = a$ as $d(a) = -2g_s l_s \log \Lambda$. By summing up all the contribution from the D0-branes, we can estimate the kinetic energy as

$$V_2 = \sum_{i=1}^{L} \sum_{m=-\tilde{n}_i}^{n_i} T_0' \, d(a+m\epsilon) = 2g_s l_s T_0' \log \left(\frac{\epsilon^{k-L}}{\Lambda^k} \prod_{i=1}^{L} n_i! \tilde{n}_i! \right), \tag{3.11}$$

where T_0' is the effective mass of the D0-brane.

Here we assume that we can use the mass of a single D0-brane as the effective mass T_0':

$$T_0' = \frac{1}{g_s l_s}. \tag{3.12}$$

Then we can write down the Boltzmann weight of this configuration as

$$Z(\tilde{\mathbf{n}}, \mathbf{n}, \epsilon, \Lambda) \equiv e^{-V_1 - V_2} = \frac{\Lambda^{2k}}{\epsilon^{2k}} \frac{\prod_{i<j}^{L} |n_i - n_j|^2 \, |\tilde{n}_i - \tilde{n}_j|^2}{\prod_{i,j=1}^{L} |n_i + \tilde{n}_j + 1|^2} \left(\frac{1}{\prod_{i=1}^{L} n_i! \tilde{n}_i!} \right)^2. \tag{3.13}$$

This is nothing but the Boltzmann weight of the Nekrasov partition function in the Frobenius representation (B.11) for $N = 1$! Note that, if the configuration $(\tilde{\mathbf{n}}, \mathbf{n})$ satisfies $\tilde{n}_i = \tilde{n}_{i+1}$ or $n_i = n_{i+1}$ for some i, the corresponding Boltzmann weight becomes zero since the potential energy (3.8) diverges. Therefore we can effectively require that $(\tilde{\mathbf{n}}, \mathbf{n})$ satisfy

$$\tilde{n}_1 > \cdots > \tilde{n}_L \geq 0, \qquad n_1 > \cdots > n_L \geq 0, \tag{3.14}$$

instead of (3.6).

3.2 $N > 1$

It is straightforward to extend the above analysis to the case of $N > 1$. We can write down a general solution of the equation (3.2) using the reducible solution (3.4);

$$\begin{aligned}
B_1^{(\tilde{\mathbf{n}}^l, \mathbf{n}^l, a_l)_{l=1}^N} &\equiv \bigoplus_{l=1}^{N} B_1^{(\tilde{\mathbf{n}}^l, \mathbf{n}^l, a_l)}, & B_2^{(\tilde{\mathbf{n}}^l, \mathbf{n}^l, a_l)_{l=1}^N} &\equiv \bigoplus_{l=1}^{N} B_2^{(\tilde{\mathbf{n}}^l, \mathbf{n}^l, a_l)}, \\
\phi^{(\tilde{\mathbf{n}}^l, \mathbf{n}^l, a_l)_{l=1}^N} &\equiv \bigoplus_{l=1}^{N} \phi^{(\tilde{\mathbf{n}}^l, \mathbf{n}^l, a_l)}, & & \\
I^{(\tilde{\mathbf{n}}^l, \mathbf{n}^l, a_l)_{l=1}^N} &\equiv \begin{pmatrix} I^{(\tilde{\mathbf{n}}_1, \mathbf{n}_1, a_1)} \\ \vdots \\ I^{(\tilde{\mathbf{n}}_N, \mathbf{n}_N, a_N)} \end{pmatrix}, & J^{(\tilde{\mathbf{n}}^l, \mathbf{n}^l, a_l)_{l=1}^N} &\equiv 0,
\end{aligned} \tag{3.15}$$

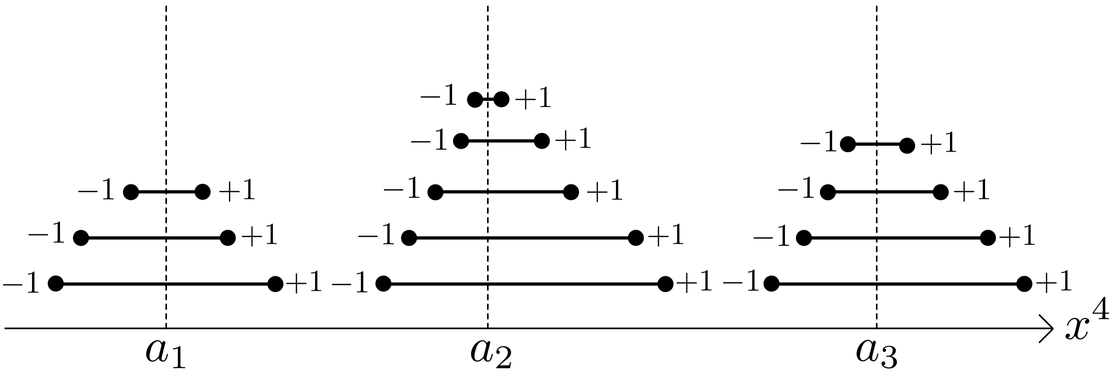

Figure 4: The configuration in the v-plane in the case of $N > 1$. We have depicted the case of $N = 3$ and $(L_1, L_2, L_3) = (3, 5, 4)$ as an example. As same as the figure 3, at the edges of each 1-dimensional object have opposite charges that make a Coulomb potential in the v-plane.

where $\tilde{\mathbf{n}}^l = (\tilde{n}^l_1, \ldots, \tilde{n}^l_{L_l})$ and $\mathbf{n}^l = (n^l_1, \ldots, n^l_{L_l})$ again satisfy (3.6) and

$$\sum_{l=1}^{N}\sum_{i=1}^{L_l}(\tilde{n}^l_i + n^l_i + 1) \equiv \sum_{l=1}^{N} k_l = k. \tag{3.16}$$

In terms of D-brane configuration, this solution corresponds to L_l spherical D2-branes around $v = a_l$ $(l = 1, \ldots, N)$; the positions of the N D4-branes. Therefore, looking at this configuration in the v-plane, there seem to be L_l 1-dimensional objects around $v = a_l$ $(l = 1, \ldots, N)$, which have opposite charges at the edges $x^4 = a_l - (\tilde{n}^l_i + 1/2)\epsilon$ and $x^4 = a_l + (n^l_i + 1/2)\epsilon$ $(i = 1, \ldots, L_l)$ (see figure 4).

We again estimate the energy of this configuration by summing up the potential energy by the two-dimensional Coulomb force in the NS5-branes and the kinetic energy of the D0-branes propagating between the NS5-branes. It is easy to see that the potential energy for the Coulomb force is given by

$$\begin{aligned} V_1 = 2\log\Bigg\{ &\prod_{l=1}^{N}\left(\epsilon^{L_l}\frac{\prod_{i,j}^{L_l}|n^l_i + \tilde{n}^l_j + 1|}{\prod_{i<j}^{L_l}|n^l_i - n^l_j||\tilde{n}^l_i - \tilde{n}^l_j|}\right) \\ &\times \prod_{l<n}^{N}\left(\prod_{i=1}^{L_l}\prod_{j=1}^{L_n}\frac{|a_{ln} + \epsilon(n^l_i + \tilde{n}^n_j + 1)||a_{ln} - \epsilon(\tilde{n}^l_i + n^n_j + 1)|}{|a_{ln} + \epsilon(n^l_i - n^n_j)||a_{ln} - \epsilon(\tilde{n}^l_i - \tilde{n}^n_j)|}\right)\Bigg\}. \end{aligned} \tag{3.17}$$

On the other hand, since the distance between the NS5-branes is now given by

$$d(v) = 2g_s l_s \sum_{l=1}^{N}\left(\log\frac{|v - a_l|}{\Lambda} - \delta_{v,a_l}\log 0\right), \tag{3.18}$$

the kinetic energy of the D0-branes becomes

$$\begin{aligned} V_2 = 2g_s l_s T'_0 \log\Bigg\{ &\prod_{l=1}^{N}\left(\frac{\epsilon^{k_l - L_l}}{\Lambda^{k_l}}\prod_{i=1}^{L_l}\tilde{n}^l_i! n^l_i!\right) \\ &\times \prod_{l=1}^{N}\prod_{n\neq l}^{N}\prod_{i=1}^{L_l}\left(|a_{ln} - \epsilon\tilde{n}^l_i| \cdot |a_{ln} - \epsilon(\tilde{n}^l_i - 1)| \cdots |a_{ln} + \epsilon n^l_i|\right)\Bigg\}. \end{aligned} \tag{3.19}$$

From (3.17) and (3.19), we obtain the Boltzmann weight of the D0-branes corresponding to the solution (3.15):

$$
\begin{aligned}
Z(\tilde{\mathbf{n}}^l, \mathbf{n}^l, \mathbf{a}, \epsilon, \Lambda) &\equiv e^{-V_1 - V_2} \\
&= \frac{\Lambda^{2kN}}{\epsilon^{2kN}} \prod_{l=1}^{N} \left\{ \frac{\prod_{i<j}^{L_l} |n_i^l - n_j^l|^2 |\tilde{n}_i^l - \tilde{n}_j^l|^2}{\prod_{i,j}^{L_l} |n_i^l + \tilde{n}_j^l + 1|^2} \left(\prod_{i=1}^{L_l} \frac{1}{n_i^l! \tilde{n}_i^l!} \right)^2 \right\} \\
&\quad \times \prod_{l<n}^{N} \Bigg\{ \prod_{i=1}^{L_l} \prod_{j=1}^{L_n} \frac{|a_{ln} + \epsilon(n_i^l - n_j^n)|^2 |a_{ln} - \epsilon(\tilde{n}_i^l - \tilde{n}_j^n)|^2}{|a_{ln} + \epsilon(n_i^l + \tilde{n}_j^n + 1)|^2 |a_{ln} - \epsilon(\tilde{n}_i^l + n_j^n + 1)|^2} \\
&\qquad \times \prod_{i=1}^{L_l} \frac{1}{|a_{ln} - \epsilon \tilde{n}_i^l|^2 \cdot |a_{ln} - \epsilon(\tilde{n}_i^l - 1)|^2 \cdots |a_{ln} + \epsilon n_i^l|^2} \\
&\qquad \times \prod_{j=1}^{L_n} \frac{1}{|a_{ln} + \epsilon \tilde{n}_j^n|^2 \cdot |a_{ln} + \epsilon(\tilde{n}_j^n - 1)|^2 \cdots |a_{ln} - \epsilon n_j^n|^2} \Bigg\}, \qquad (3.20)
\end{aligned}
$$

under the assumption (3.12). This expression again coincides with the instanton part of the Nekrasov partition function in the Frobenius representation (B.11). From this result, we conclude that the Nekrasov partition function is that of D0-branes bound to D4-branes in the presence of NSNS B-field (2.14) and RR 3-form (2.17) in the background of the Hanany-Witten type brane configuration.

4. Nekrasov partition function for $\mathcal{N} = 2$ QCD from D0-branes

In this section, we reproduce the Nekrasov partition function for four-dimensional $\mathcal{N} = 2$ theory with hypermultiplets in the fundamental representations [3] as a non-trivial check of the method developed in the previous section.

In order to introduce matter fields in the fundamental representation, we add N_f semi-infinite D4-branes attached to one of the NS5-branes at $v = -m_1, \ldots, -m_{N_f}$ in addition to the N D4-branes stretched between the NS5-branes (figure 5). The positions of these D4-branes correspond to the bare masses of the hypermultiplets.

The strategy to construct the partition function of D0-branes is the same with the previous section, namely we use as the same configuration of D0-branes $\{(\tilde{\mathbf{n}}_l, \mathbf{n}_l, a_l) \,|\, l = 1, \ldots, N\}$ as is used in the previous section. Then the potential energy for the Coulomb force in the v-plane is the same with (3.17). However, the kinetic energy is different from (3.19) since the NS5-branes are further deformed from (2.3) by the presence of additional D4-branes:

$$
\begin{aligned}
\tau_- &= -g_s l_s \sum_{l=1}^{N} \log\left(\frac{|v - a_l|}{\Lambda}\right), \\
\tau_+ &= g_s l_s \left[\sum_{l=1}^{N} \log\left(\frac{|v - a_l|}{\Lambda}\right) - \sum_{f=1}^{N_f} \log\left(\frac{|v + m_f|}{\Lambda}\right) \right], \qquad (4.1)
\end{aligned}
$$

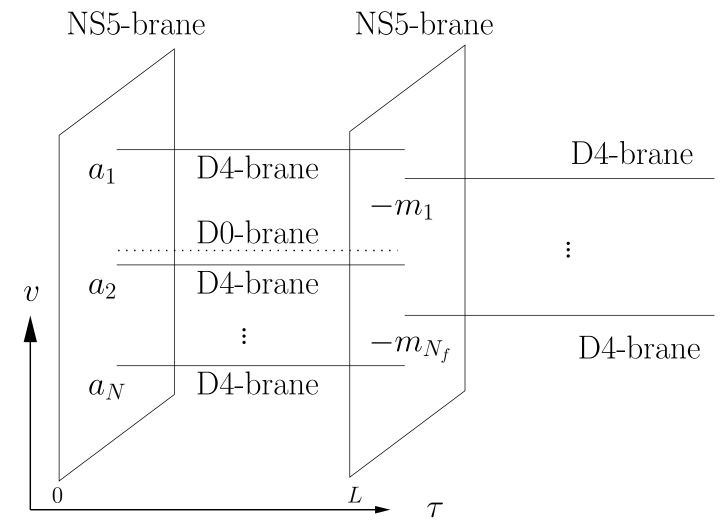

Figure 5: The brane configuration to realize four-dimensional $\mathcal{N}=2$ supersymmetric gauge theory with hypermultiplets in the fundamental representation. In addition to figure 1, we have added N_f D4-branes that attach to one of the NS5-branes at $v=-m_1,\ldots,-m_{N_f}$. They have semi-infinite world-volumes in the direction τ.

where $\tau_\pm$ are the positions of the NS5-branes in the direction τ. Then, we see that the distance between the the NS5-branes at v is given by

$$d(v)=g_s l_s\left[2\sum_{l=1}^{N}\log\left(\frac{|v-a_l|}{\Lambda}\right)-\sum_{f=1}^{N_f}\log\left(\frac{|v+m_f|}{\Lambda}\right)\right], \tag{4.2}$$

and the kinetic energy of the D0-branes can be estimated as

$$V_2=g_s l_s T_0'\log\left\{\prod_{l=1}^{N}\left(\frac{\epsilon^{k_l-L_l}}{\Lambda^{k_l}}\prod_{i=1}^{L_l}\tilde{n}_i^l!n_i^l!\right)^2\times\prod_{l=1}^{N}\prod_{n\neq l}^{N}\prod_{i=1}^{L_l}\Big(|a_{ln}-\epsilon\tilde{n}_i^l|\cdots|a_{ln}+\epsilon n_i^l|\Big)^2\right.$$
$$\left.\times\Lambda^{kN_f}\prod_{f=1}^{N_f}\prod_{l=1}^{N}\prod_{i=1}^{L_l}\left(\frac{1}{|m_f+a_l-\epsilon\tilde{n}_i^l|\cdots|m_f+a_l+\epsilon n_i^l|}\right)\right\}. \tag{4.3}$$

Combining (3.17) and (4.3) and assuming (3.12), we obtain the Boltzmann weight corresponding to this configuration of D0-branes;

$$Z(\tilde{\mathbf{n}}^l,\mathbf{n}^l,\mathbf{a},\mathbf{m},\epsilon,\Lambda)=Z(\tilde{\mathbf{n}}^l,\mathbf{n}^l,\mathbf{a},\epsilon,\Lambda)$$
$$\times\frac{1}{\Lambda^{kN_f}}\prod_{f=1}^{N_f}\prod_{l=1}^{N}\prod_{i=1}^{L_l}\Big(|m_f+a_l-\epsilon\tilde{n}_i^l|\cdots|m_f+a_l+\epsilon n_i^l|\Big), \tag{4.4}$$

where $Z(\tilde{\mathbf{n}}^l,\mathbf{n}^l,\mathbf{a},\epsilon,\Lambda)$ is given by (3.20). Looking at (C.6), we see that (4.4) coincides with the instanton part of the Nekrasov partition function for four-dimensional $\mathcal{N}=2$ theory with N_f hypermultiplets in the fundamental representation.

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5. Conclusion and discussion

In this paper, we analyzed the behavior of D0-branes in the Hanany-Witten type brane configuration in a background of RR 4-form field strength and NSNS B-field. We showed that the partition function of Euclidean D0-branes propagating between the NS5-branes coincides with the Nekrasov partition function of instantons in four-dimensional $\mathcal{N}=2$ supersymmetric Yang-Mills theory. In this analysis, the Myers effect played an important role. We applied the same method to the brane configuration realizing four-dimensional $\mathcal{N}=2$ theory QCD and the partition function of the D0-branes again coincides with the Nekrasov partition function of the theory.

There would be many applications in the method developed in this paper. As a straightforward application, we can apply it to $\mathcal{N}=2$ quiver gauge theories and/or $\mathcal{N}=2$ theories with other gauge groups than SU(N) [60]. It would also be interesting to reduce supersymmetry from $\mathcal{N}=2$ to $\mathcal{N}=1$ by deforming the NS5-branes holomorphically, which might give a connection to Dijkgraaf-Vafa theory [61 – 63]. Although we have concentrated on four-dimensional theories in this paper, a five-dimensional version of the instanton partition function is also proposed [3]. In terms of the brane configuration we have used in this paper, this would be achieved by lifting it up to a configuration of a M5-brane in the background of a 4-form field strength in the M-theory. It is interesting to extend our analysis to the M-theory and see how the five-dimensional version of the instanton partition function appears.

Lastly, as mentioned in Introduction, the Nekrasov partition function is known to be equivalent to amplitudes of topological string theory of local toric Calabi-Yau manifolds [21 – 26]. In the heart of this relation, there is an idea of geometrical engineering [64, 65]; by realizing 4D $\mathcal{N}=2$ gauge theory by compactifying Type II superstring theory by a Calabi-Yau three-fold, some nature of the gauge theory is explained as a geometrical property of the Calabi-Yau manifold. It is interesting that the same partition function is obtained from rather simple set-up of branes in a RR-background. From this result, it would be quite natural to expect that the brane system with RR flux would be connected to a Calabi-Yau set-up by a sequence of string duality. It would be an important and interesting future work to reveal this connection.

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A. Frobenius representation of Young diagram

In this appendix, we introduce the Frobenius representation of Young diagram.

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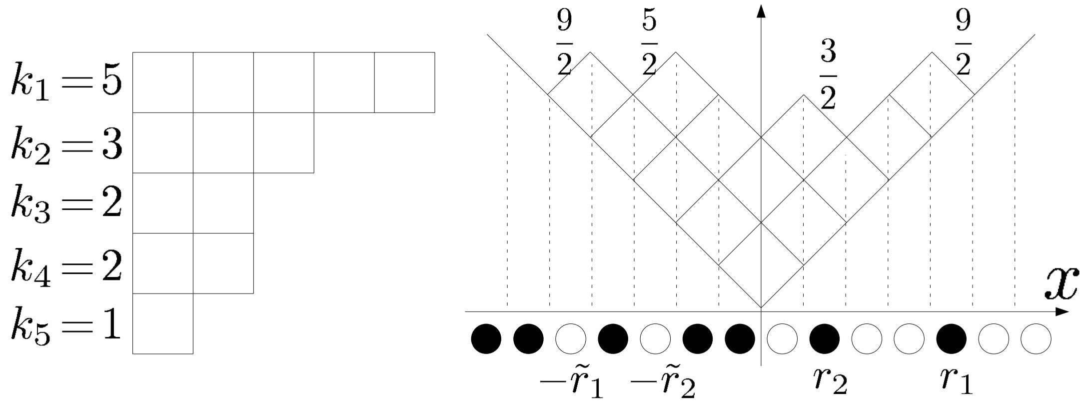

Figure 6: An example of Young diagram in the Frobenius representation. The Young diagram in the left figure is expressed by the partition $\mathbf{k} = \{5, 3, 2, 2, 1\}$. The right figure is the corresponding profile function. We see that the Frobenius representation of this Young diagram is given by $(9/2, 5/2 \,|\, 9/2, 3/2)$. We also see that there are white circles at $x = -9/2$ and $-5/2$ and there are black circles at $x = 9/2$ and $3/2$, which is the corresponding Maya diagram.

We start with a Young diagram parametrized by $\{k_i\}$, the number of boxes in the i's row satisfying

$$k_1 + \cdots + k_r = k, \qquad k_1 \geq k_2 \geq \cdots \geq k_r > 0, \tag{A.1}$$

which is a partition of the integer k. So we can identify the Young diagram with the partition $\mathbf{k} = \{k_1, \ldots, k_r\}$ itself.

For our purpose to write down the Nekrasov partition function, it is useful to draw the diagram by inclining 45 degrees. Here we divide the (inclined) diagram into two parts by the center line (figure 6). Suppose there are L boxes on this line. Let $\tilde{r}_i$ and r_i $(i = 1, \ldots, L)$ denote the numbers of boxes in the left and right of the i's box on the center, respectively. By counting the "number" of the center box in the left and right as $1/2$, respectively, we can express the Young diagram $\mathbf{k}$ by a set of half integers $\tilde{r}_i$ and r_i:

$$(\tilde{\mathbf{r}}|\mathbf{r}) = (\tilde{r}_1, \ldots, \tilde{r}_L | r_1, \ldots, r_L), \qquad \tilde{r}_i,\, r_i \in \mathbb{N} - 1/2. \tag{A.2}$$

This expression is called the Frobenius representation of the Young diagram (or the partition) $\mathbf{k}$.

Incidentally, this representation is deeply related with the so-called Maya diagram, which is defined as a sequence of black and white circles on a line. We start the situation where the black circles and the white circles are at $x \in -\mathbb{N} + 1/2$ and $x \in \mathbb{N} - 1/2$, respectively. Maya diagram is obtained by exchanging the positions of arbitrary pairs of black and white circles. By construction, the number of white circles in the region $x < 0$ and that of black circles in the region $x > 0$ is the same. In the Frobenius representation (A.2), $-\tilde{r}_i$ and r_i are identified with the positions of white circles in $x < 0$ and black circles in $x > 0$, respectively. We draw an example of these relations in figure 6.

B. Nekrasov partition function in the Frobenius representation

In this appendix, we review the Nekrasov partition function (for $\epsilon_1 = -\epsilon_2 \equiv \epsilon$) and rewrite it in the Frobenius representation.

The Nekrasov partition function is given by

$$Z_{\mathrm{Nek}}\,(\boldsymbol{a},\epsilon,\Lambda) = Z_{\mathrm{pert}}(\boldsymbol{a},\epsilon)\sum_{k=1}^{\infty}\sum_{\substack{k_1,\ldots,k_N\in\mathbb{Z}_{\geq 0}\\ k_1+\cdots+k_N=k}}\sum_{\vec{k}_1\in Y_{k_1}}\cdots\sum_{\vec{k}_N\in Y_{k_N}} Z_{\mathrm{inst}}(\boldsymbol{a},\mathbf{k},\epsilon,\Lambda), \tag{B.1}$$

with

$$Z_{\mathrm{pert}}(\boldsymbol{a},\epsilon) = \exp\left\{\sum_{l\neq n}\gamma_\epsilon(a_l-a_n;\Lambda)\right\}, \tag{B.2}$$

$$Z_{\mathrm{inst}}(\boldsymbol{a},\mathbf{k},\epsilon,\Lambda) = \Lambda^{2Nk}\prod_{(l,i)\neq(n,j)}\frac{a_{ln}+\epsilon(k_{l,i}-k_{n,j}+j-i)}{a_{ln}+\epsilon(j-i)}, \tag{B.3}$$

where $i,j = 1,\ldots,\infty$, $l,n = 1,\ldots,N$, $a_{ln} \equiv a_l - a_n$, $\gamma_\epsilon(x;\Lambda)$ is defined through the deference equation,

$$\gamma_\epsilon(x+\epsilon;\Lambda)+\gamma_\epsilon(x-\epsilon;\Lambda)-2\gamma_\epsilon(x;\Lambda) = \log(x/\Lambda), \tag{B.4}$$

and Y_l is a set of Young diagrams with l boxes, whose element is given as a partition of l, namely $\vec{l} = (l_1,l_2,\ldots)$ satisfying $l_1+l_2+\cdots = l$ with $l_1\geq\cdots\geq l_r > l_{r+1} = l_{r+2}\cdots = 0$.

As shown in [10], the Nekrasov partition function can be compactly expressed using a piecewise-linear function called the (colored) profile function,

$$f_{\mathbf{a},\mathbf{k}}(x|\epsilon) \equiv \sum_{l=1}^{N} f_{\mathbf{k}_l}(x-a_l|\epsilon), \tag{B.5}$$

with

$$f_{\mathbf{k}}(x|\epsilon) \equiv |x| + \sum_{i=1}^{\infty}\left[|x-\epsilon(k_i-i+1)| - |x-\epsilon(k_i-i)) - |x-\epsilon(-i+1)| + |x-\epsilon(-i)|\right], \tag{B.6}$$

where $\mathbf{k}_l$ are Young diagrams sitting at $x = a_l$. Using (B.5), the instanton part of the Nekrasov partition function (B.3) can be written as

$$Z_{\mathrm{inst}}(\boldsymbol{a},\mathbf{k},\epsilon,\Lambda) = \exp\left(-\frac{1}{4}\int_{x\neq y} dxdy\, f''_{\mathbf{a},\mathbf{k}}(x|\epsilon) f''_{\mathbf{a},\mathbf{k}}(y|\epsilon)\gamma_\epsilon(x-y;\Lambda)\right). \tag{B.7}$$

We can rewrite the profile function (B.6) using the Frobenius representation (A.2). To this end, we express the Young diagrams $\mathbf{k}_l$ by $(\tilde{\mathbf{r}}_l|\mathbf{r}_l)$ $(l = 1,\ldots,N)$ and introduce the integers,

$$\tilde{n}^l_i \equiv \tilde{r}^l_i - 1/2, \quad n^l_i \equiv r^l_i - 1/2, \qquad (i = 1,\ldots,L_l) \tag{B.8}$$

which satisfy

$$\tilde{n}^l_1 > \cdots > \tilde{n}^l_{L_l} \geq 0, \qquad n^l_1 > \cdots > n^l_{L_l} \geq 0. \tag{B.9}$$

Using this, the second derivative of the profile function can be rewritten as

$$f''_{\mathbf{k}}(x|\epsilon) = \frac{1}{2}\left[\delta(x)+\sum_{i=1}^{L}\Big(-\delta(x+\epsilon(\tilde{n}_i+1))+\delta(x+\epsilon\tilde{n}_i)+\delta(x-\epsilon n_i)-\delta(x-\epsilon(n_i+1))\Big)\right]. \tag{B.10}$$

Substituting this expression into (B.7), we can rewrite the instanton part of the partition function (B.3) as

$$\begin{aligned} Z_{\text{inst}}(\boldsymbol{a},\mathbf{k},\epsilon,\Lambda) = \frac{\Lambda^{2kN}}{\epsilon^{2kN}}\prod_{l=1}^{N}&\left\{\frac{\prod_{i<j}^{L_l}|n^l_i-n^l_j|^2|\tilde{n}^l_i-\tilde{n}^l_j|^2}{\prod_{i,j}^{L_l}|n^l_i+\tilde{n}^l_j+1|^2}\left(\prod_{i=1}^{L_l}\frac{1}{n^l_i!\tilde{n}^l_i!}\right)^2\right\}\\ &\times\prod_{l<n}^{N}\left\{\prod_{i=1}^{L_l}\prod_{j=1}^{L_n}\frac{|a_{ln}+\epsilon(n^l_i-n^n_j)|^2|a_{ln}-\epsilon(\tilde{n}^l_i-\tilde{n}^n_j)|^2}{|a_{ln}+\epsilon(n^l_i+\tilde{n}^n_j+1)|^2|a_{ln}-\epsilon(\tilde{n}^l_i+n^n_j+1)|^2}\right.\\ &\left.\times\prod_{i=1}^{L_l}\frac{1}{|a_{ln}-\epsilon\tilde{n}^l_i|^2\cdots|a_{ln}+\epsilon n^l_i|^2}\times\prod_{j=1}^{L_n}\frac{1}{|a_{ln}+\epsilon\tilde{n}^n_j|^2\cdots|a_{ln}-\epsilon n^n_j|^2}\right\}. \end{aligned} \tag{B.11}$$

C. Nekrasov partition function for four-dimensional $\mathcal{N}=2$ QCD

In this appendix, we summarize the Nekrasov partition function for four-dimensional $\mathcal{N}=2$ theory with hypermultiplets in the fundamental representation and rewrite it using the Frobenius representation.

Let $\mathbf{m}$ denote the vector of bare masses of the hypermultiplets:

$$\mathbf{m} = (m_1,\ldots,m_{N_f}). \tag{C.1}$$

Then the Nekrasov partition function is given by [3]

$$Z^{\text{f}}_{\text{Nek}}(\boldsymbol{a},\mathbf{m},\epsilon,\Lambda) = Z^{\text{f}}_{\text{pert}}(\boldsymbol{a},\mathbf{m},\epsilon,\Lambda)\sum_{k=1}^{\infty}\sum_{\substack{k_1,\ldots,k_N\in\mathbb{Z}_{\geq 0}\\ k_1+\cdots+k_N=k}}\sum_{\vec{k}_1\in Y_{k_1}}\cdots\sum_{\vec{k}_N\in Y_{k_N}} Z^{\text{f}}_{\text{inst}}(\boldsymbol{a},\mathbf{m},\mathbf{k},\epsilon,\Lambda), \tag{C.2}$$

with

$$Z^{\text{f}}_{\text{pert}}(\boldsymbol{a},\epsilon,\Lambda) = \exp\left\{\sum_{l\neq n}\gamma_\epsilon(a_l-a_n;\Lambda)+\sum_{l,f}\gamma_\epsilon(a_l+m_f;\Lambda)\right\}, \tag{C.3}$$

$$\begin{aligned} Z^{\text{f}}_{\text{inst}}(\boldsymbol{a},\mathbf{m},\mathbf{k},\epsilon,\Lambda) = \Lambda^{k(2N-N_f)}&\prod_{(l,i)\neq(n,j)}\frac{a_{ln}+\epsilon(k_{l,i}-k_{n,j}+j-i)}{a_{ln}+\epsilon(j-i)}\\ &\times\prod_{l,f,i}\frac{\Gamma\left(\frac{m_f+a_l}{\epsilon}+k_{li}-i+1\right)}{\Gamma\left(\frac{m_f+a_l}{\epsilon}-i+1\right)}, \end{aligned} \tag{C.4}$$

where $i, j = 1, \ldots, \infty$, $l, n = 1, \ldots, N$, $f = 1, \ldots, N_f$, $a_{ln} \equiv a_l - a_n$, $\gamma_\epsilon(x; \Lambda)$ is defined in (B.4), and Y_l is again a set of Young diagrams with l boxes. In [10], it was shown that (C.4) can be written using the colored profile function (B.5) as

$$Z^{\mathrm{f}}_{\mathrm{inst}}(\boldsymbol{a}, \mathbf{m}, \mathbf{k}, \epsilon, \Lambda) = \exp\Bigg(-\frac{1}{4}\int_{x\neq y} dxdy f''_{\mathbf{a},\mathbf{k}}(x|\epsilon) f''_{\mathbf{a},\mathbf{k}}(y|\epsilon)\gamma_\epsilon(x-y;\Lambda) \\ + \frac{1}{2}\sum_{f=1}^{N_f}\int dx f''_{\mathbf{a},\mathbf{k}}(x|\epsilon)\gamma_\epsilon(x+m_f;\Lambda)\Bigg). \quad \text{(C.5)}$$

The easiest way to rewrite (C.4) in the Frobenius representation is substituting (B.10) into the expression (C.5). The result is

$$Z^{\mathrm{f}}_{\mathrm{inst}}(\boldsymbol{a}, \mathbf{m}, \mathbf{k}, \epsilon, \Lambda) = Z_{\mathrm{inst}}(\mathbf{a}, \mathbf{k}, \epsilon, \Lambda) \\ \times \frac{1}{\Lambda^{kN_f}} \prod_{f=1}^{N_f}\prod_{l=1}^{N}\prod_{i=1}^{L_l}\Big(|m_f + a_l - \epsilon\tilde{n}^l_i| \cdots |m_f + a_l + \epsilon n^l_i|\Big), \quad \text{(C.6)}$$

where $Z_{\mathrm{inst}}(\mathbf{a}, \mathbf{k}, \epsilon, \Lambda)$ is given by (B.11). In deriving (C.6), we have used the relation,

$$\gamma_\epsilon(x+\epsilon;\Lambda) - \gamma_\epsilon(x;\Lambda) = \log\left(\frac{\epsilon^{x/\epsilon}}{\sqrt{2\pi}}\Gamma(x/\epsilon+1)\right). \quad \text{(C.7)}$$

References

[1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [*Erratum ibid.* **430** (1994) 485] [hep-th/9407087].

[2] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099].

[3] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [hep-th/0206161].

[4] A.S. Losev, A. Marshakov and N.A. Nekrasov, *Small instantons, little strings and free fermions*, hep-th/0302191.

[5] E. Witten, *Two-dimensional gauge theories revisited*, *J. Geom. Phys.* **9** (1992) 303 [hep-th/9204083].

[6] S. Cordes, G.W. Moore and S. Ramgoolam, *Lectures on 2D Yang-Mills theory, equivariant cohomology and topological field theories*, *Nucl. Phys.* **41** *(Proc. Suppl.)* (1995) 184 [hep-th/9411210].

[7] G.W. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs branches*, *Commun. Math. Phys.* **209** (2000) 97 [hep-th/9712241].

[8] G.W. Moore, N. Nekrasov and S. Shatashvili, *D-particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77 [hep-th/9803265].

[9] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, *JHEP* **05** (2003) 054 [hep-th/0211108].

JHEP09(2008)083

[10] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, hep-th/0306238.

[11] H. Nakajima and K. Yoshioka, *Instanton counting on blowup. I*, math.AG/0306198.

[12] H. Nakajima and K. Yoshioka, *Lectures on instanton counting*, math.AG/0311058.

[13] T. Maeda, T. Nakatsu, K. Takasaki and T. Tamakoshi, *Free fermion and Seiberg-Witten differential in random plane partitions*, *Nucl. Phys.* **B 715** (2005) 275 [hep-th/0412329].

[14] M. Billó, M. Frau, F. Fucito and A. Lerda, *Instanton calculus in RR background and the topological string*, *JHEP* **11** (2006) 012 [hep-th/0606013].

[15] R. Blumenhagen, M. Cvetič, R. Richter and T. Weigand, *Lifting D-instanton zero modes by recombination and background fluxes*, *JHEP* **10** (2007) 098 [arXiv:0708.0403].

[16] K. Ito, H. Nakajima and S. Sasaki, *Deformation of super Yang-Mills theories in RR* 3*-form background*, *JHEP* **07** (2007) 068 [arXiv:0705.3532].

[17] M. Billó et al., *Instantons in $N = 2$ magnetized D-brane worlds*, *JHEP* **10** (2007) 091 [arXiv:0708.3806].

[18] S. Sasaki, K. Ito and H. Nakajima, *Instantons in deformed super Yang-Mills theories*, arXiv:0710.2218.

[19] M. Billó et al., *Flux interactions on D-branes and instantons*, arXiv:0807.1666.

[20] M. Billó et al., *Non-perturbative effective interactions from fluxes*, arXiv:0807.4098.

[21] A. Iqbal and A.-K. Kashani-Poor, *Instanton counting and Chern-Simons theory*, *Adv. Theor. Math. Phys.* **7** (2004) 457 [hep-th/0212279].

[22] A. Iqbal and A.-K. Kashani-Poor, SU(N) *geometries and topological string amplitudes*, *Adv. Theor. Math. Phys.* **10** (2006) 1 [hep-th/0306032].

[23] T. Eguchi and H. Kanno, *Topological strings and Nekrasov's formulas*, *JHEP* **12** (2003) 006 [hep-th/0310235].

[24] T. Eguchi and H. Kanno, *Geometric transitions, Chern-Simons gauge theory and Veneziano type amplitudes*, *Phys. Lett.* **B 585** (2004) 163 [hep-th/0312234].

[25] T.J. Hollowood, A. Iqbal and C. Vafa, *Matrix models, geometric engineering and elliptic genera*, *JHEP* **03** (2008) 069 [hep-th/0310272].

[26] J. Zhou, *Curve counting and instanton counting*, math.AG/0311237.

[27] M. Aganagic, M. Mariño and C. Vafa, *All loop topological string amplitudes from Chern-Simons theory*, *Commun. Math. Phys.* **247** (2004) 467 [hep-th/0206164].

[28] A. Iqbal, *All genus topological string amplitudes and* 5*-brane webs as Feynman diagrams*, hep-th/0207114.

[29] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, *The topological vertex*, *Commun. Math. Phys.* **254** (2005) 425 [hep-th/0305132].

[30] A. Okounkov, N. Reshetikhin and C. Vafa, *Quantum Calabi-Yau and classical crystals*, hep-th/0309208.

[31] A. Iqbal, N. Nekrasov, A. Okounkov and C. Vafa, *Quantum foam and topological strings*, *JHEP* **04** (2008) 011 [hep-th/0312022].

JHEP09(2008)083

[32] Y. Tachikawa, *Five-dimensional Chern-Simons terms and Nekrasov's instanton counting*, *JHEP* **02** (2004) 050 [hep-th/0401184].

[33] H. Awata and H. Kanno, *Instanton counting, Macdonald functions and the moduli space of D-branes*, *JHEP* **05** (2005) 039 [hep-th/0502061].

[34] S. Matsuura and K. Ohta, *Localization on the D-brane, two-dimensional gauge theory and matrix models*, *Phys. Rev.* **D 73** (2006) 046006 [hep-th/0504176].

[35] T. Maeda, T. Nakatsu, Y. Noma and T. Tamakoshi, *Gravitational quantum foam and supersymmetric gauge theories*, *Nucl. Phys.* **B 735** (2006) 96 [hep-th/0505083].

[36] A. Marshakov and N. Nekrasov, *Extended Seiberg-Witten theory and integrable hierarchy*, *JHEP* **01** (2007) 104 [hep-th/0612019].

[37] T. Maeda and T. Nakatsu, *Amoebas and instantons*, *Int. J. Mod. Phys.* **A 22** (2007) 937 [hep-th/0601233].

[38] A. Iqbal, C. Kozcaz and C. Vafa, *The refined topological vertex*, hep-th/0701156.

[39] A. Marshakov, *On microscopic origin of integrability in Seiberg-Witten theory*, *Theor. Math. Phys.* **154** (2008) 362 [arXiv:0706.2857].

[40] M. Taki, *Refined topological vertex and instanton counting*, *JHEP* **03** (2008) 048 [arXiv:0710.1776].

[41] T. Nakatsu and K. Takasaki, *Melting crystal, quantum torus and Toda hierarchy*, arXiv:0710.5339.

[42] A. Marshakov, *Seiberg-Witten theory and extended Toda hierarchy*, *JHEP* **03** (2008) 055 [arXiv:0712.2802].

[43] H. Awata and H. Kanno, *Refined BPS state counting from Nekrasov's formula and Macdonald functions*, arXiv:0805.0191.

[44] T. Nakatsu, Y. Noma and K. Takasaki, *Integrable structure of* 5d $\mathcal{N}=1$ *supersymmetric Yang-Mills and melting crystal*, *Int. J. Mod. Phys.* **A 23** (2008) 2332 [arXiv:0806.3675].

[45] T. Nakatsu, Y. Noma and K. Takasaki, *Extended* 5d *Seiberg-Witten theory and melting crystal*, arXiv:0807.0746.

[46] S. de Haro and M. Tierz, *Brownian motion, Chern-Simons theory and* 2d *Yang-Mills*, *Phys. Lett.* **B 601** (2004) 201 [hep-th/0406093].

[47] T. Matsuo, S. Matsuura and K. Ohta, *Large-N limit of* 2*D Yang-Mills theory and instanton counting*, *JHEP* **03** (2005) 027 [hep-th/0406191].

[48] T.-S. Tai, *Instanton counting and matrix model*, *Prog. Theor. Phys.* **119** (2008) 165 [arXiv:0709.0432].

[49] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[50] E. Witten, *Solutions of four-dimensional field theories via M-theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166].

[51] R.C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022 [hep-th/9910053].

[52] A. Strominger, *Open p-branes*, *Phys. Lett.* **B 383** (1996) 44 [hep-th/9512059].

[53] A. Giveon and D. Kutasov, *Brane dynamics and gauge theory*, *Rev. Mod. Phys.* **71** (1999) 983 [hep-th/9802067].

[54] M. Berkooz and M.R. Douglas, *Five-branes in M(atrix) theory*, *Phys. Lett.* **B 395** (1997) 196 [hep-th/9610236].

[55] M. Van Raamsdonk, *Open dielectric branes*, *JHEP* **02** (2002) 001 [hep-th/0112081].

[56] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *The calculus of many instantons*, *Phys. Rept.* **371** (2002) 231 [hep-th/0206063].

[57] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142].

[58] H.W. Braden and N.A. Nekrasov, *Space-time foam from non-commutative instantons*, *Commun. Math. Phys.* **249** (2004) 431 [hep-th/9912019].

[59] N. Nekrasov and A.S. Schwarz, *Instantons on noncommutative R^4 and $(2,0)$ superconformal six dimensional theory*, *Commun. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[60] K.J. Larsen and S. Matsuura, in preparation.

[61] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[62] R. Dijkgraaf and C. Vafa, *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106].

[63] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[64] S.H. Katz, A. Klemm and C. Vafa, *Geometric engineering of quantum field theories*, *Nucl. Phys.* **B 497** (1997) 173 [hep-th/9609239].

[65] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of $4D$ $N = 2$ gauge theories. I*, *Adv. Theor. Math. Phys.* **1** (1998) 53 [hep-th/9706110].

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